AXIAL LAMINAR FLOW OF A VISCOPLASTIC FLUID IN AN ANNULAR TUBE

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Approximate relations suitable for engineering calculations are derived for determining the boundary of the core when the pressure gradient is given, the boundary and width of the core when the volumetric flow rate is given, and the pressure losses.

The axial flow of a viscoplastic fluid in the annular space between two coaxial cylinders has been investigated by a number of authors [1-8]. The problem was first solved by Volarovich and Gutkin in [1]. Analytical expressions were obtained for the velocity profiles and for the volumetric flow rate, with a constant of integration. The calculation of the latter required the solution of a transcendental equation. For the case in which the thickness of the gap is less than the radius of the inside cylinder the authors gave a first-approximation relation for calculating the root of the transcendental equation. In [2] Shchipanov found the second approximation. Elaborating their own work, Volarovich and Gutkin analyzed the boundary-layer flow regime when the thickness of the flow zone is much smaller than the thickness of the core [8]. For the velocity profiles and volumetric flow rate in this flow regime they obtained relations that are valid for any ratio of tube radii.

An approximate relation for the flow of fluid through an annular capillary is given in [3] without derivation. The author does not indicate the limits of applicability of his relation, but, if we are to judge from its form, it was derived on the assumption of a small core. The flow problem for a viscoplastic fluid in an annular tube has been investigated by Mori and Ototake [4], but their solution is erroneous. Laird [5] obtained a solution similar to that of Volarovich and Gutkin, emulating their method of solution. The complete expression that they obtained for the flow rate is rather cumbersome. For small cores they derived an approximate relation. Concurrently with Laird, Slibar and Paslay [6] analyzed the flow problem for a viscoplastic fluid in an annular tube. Unlike [1, 5], in which ordinary second-order differential equations were integrated in order to determine the velocity profile in the shear-flow domains, in [6] the authors first found a unified shear-stress distribution for the entire annular gap. The velocity profiles in the shear-flow domains were then determined by integration of the rheological equation relating the shear stress to the velocity gradient.

Fredrickson and Bird [7] undertook the task of deriving a simpler expression for the volumetric flow rate and presented their solution in a compact form, using dimensionless variables. Their paper includes exhaustive tables of values of the outside radius of the core and the core velocity for various ratios of the cylinder radii and core widths. The results of the calculations are also summarized in graphical form. An approximate relation was derived for small cores.

The principal difficulty in trying to solve the given problem lies in the determination of the constant of integration (core boundary) from the transcendental equation when the pressure gradient is given or, when the volumetric flow rate is given, in the determination of the core boundary and core width from a set of two equations, one of which is transcendental and the other is a fourth-degree algebraic equation. The lack of analytical expressions for the roots of the transcendental equation and the set of equations greatly impedes the analysis of the solution. In particular, it is difficult to use the solution for the processing of viscosimetric data obtained by means of a capillary with coaxial tubes, or to obtain such an important characteristic as the pressure losses associated with the axial flow of a viscoplastic fluid in an

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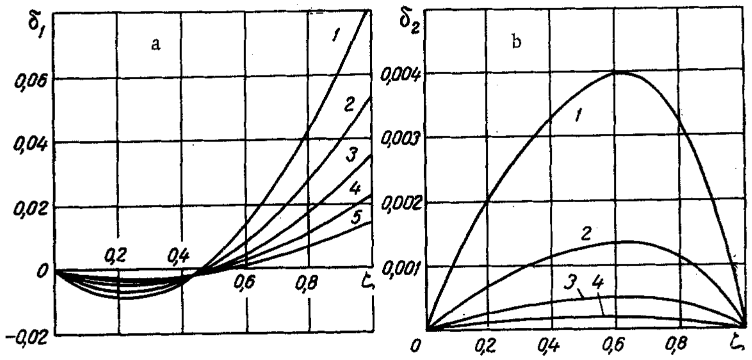

Fig. 1. Absolute errors δ_1 (a) and δ_2 (b) versus the relative core width ζ. 1) $\varkappa = 0.1$; 2) 0.2; 3) 0.3; 4) 0.4; 5) 0.5.

annular tube. In some of the papers cited above approximate relations have been obtained for the volumetric flow rate under certain constraints, mainly the assumption of a narrow core, but in none of those papers has the error of the resulting equation been estimated.

We now give approximate expressions, suitable for engineering calculations, for the roots of the transcendental equation and roots of the combined transcendental and algebraic equations; these expressions are valid for all core widths and for a wide range of ratios of the cylinder radii.

1. Suppose that a viscoplastic fluid with a shear strength τ_0 and plastic viscosity μ flows under the action of a pressure gradient $P = dp/dz$ in the annular gap between two coaxial circular cylinders of radius R_1 and R_2 ($R_1 < R_2$). We use the solution obtained by Fredrickson and Bird [7] as the most compact and amenable to investigation. The dimensionless velocity profiles v_1 and v_2 in the shear-flow domains, the core velocity v_0, and the volumetric flow rate Q are expressed by the relations

$$v_1(\rho) = \beta(\varkappa - \rho) - \frac{1}{2}(\rho^2 - \varkappa^2) + \rho_2(\rho_2 - \beta)\ln(\rho/\varkappa), \quad \varkappa \leqslant \rho \leqslant \rho_1, \tag{1.1}$$

$$v_2(\rho) = \beta(\rho - 1) + \frac{1}{2}(1 - \rho^2) + \rho_2(\rho_2 - \beta)\ln\rho, \quad \rho_2 \leqslant \rho \leqslant 1, \tag{1.2}$$

$$v_0(\rho) = v_1(\rho_1) = v_2(\rho_2), \quad \rho_1 \leqslant \rho \leqslant \rho_2, \tag{1.3}$$

$$Q = \frac{\pi R_2^4 P}{8\mu}\left[1 - \varkappa^4 - 2\rho_2(\rho_2 - \beta)(1 - \varkappa^2) - \frac{4}{3}(1 + \varkappa^3)\beta + \frac{1}{3}(2\rho_2 - \beta)^3\beta\right]. \tag{1.4}$$

Here

$$v = \frac{2\mu}{PR_2^2}v_z; \; \rho = \frac{r}{R_2}; \; \beta = \frac{2\tau_0}{PR_2}; \; \varkappa = \frac{R_1}{R_2}. \tag{1.5}$$

The outside and inside radii ρ_2 and ρ_1 of the core are related by the equation

$$\rho_2 - \rho_1 = \beta, \tag{1.6}$$

i.e., the parameter β expresses the dimensionless width of the core. If the pressure gradient P is given, then the parameter β is known, and for the complete solution of the problem it is required to know the outside radius ρ_2 of the core. The latter is determined from the transcendental equation

$$2\rho_2(\rho_2 - \beta)\ln\frac{\rho_2 - \beta}{\rho_2\varkappa} - 1 + (\beta + \varkappa)^2 + 2\beta(1 - \rho_2) = 0, \tag{1.7}$$

which is obtained from the condition of equal velocities at the boundaries of the core. We now find an approximate expression for the roots of Eq. (1.7).

It is clear from (1.5) that as $\tau_0 \to 0$ the core width β tends to zero, and in the limit $\beta = 0$ (Newtonian fluid) we obtain from (1.7)

$$\lambda_0 = \sqrt{\frac{\varkappa^2 - 1}{2\ln\varkappa}} \tag{1.8}$$

TABLE 1. Maximum Values $\delta_2^*(\varkappa)$ of the Absolute Errors for Various Values of $\varkappa$

$\varkappa$	δ_2^*	$\varkappa$	δ_2^*	$\varkappa$	δ_2^*
0,1	$0,399\cdot10^{-2}$	0,4	$0,203\cdot10^{-3}$	0,7	$0,619\cdot10^{-5}$
0,2	$0,136\cdot10^{-2}$	0,5	$0,745\cdot10^{-4}$	0,8	$0,102\cdot10^{-5}$
0,3	$0,522\cdot10^{-3}$	0,6	$0,241\cdot10^{-4}$	0,9	$0,541\cdot10^{-7}$

(λ_0 is the value of ρ_2 when $\beta = 0$). It is expected that for small β the values of $\rho_2^{(0)}$ calculated according to the relation

$$\rho_2^{(0)} = \lambda_0 + \frac{\beta}{2}, \tag{1.9}$$

should be close to the roots of the transcendental equation (1.7). Curves of the absolute error $\delta_1 = \rho_2 - \rho_2^{(0)}$ as a function of the dimensionless relative core width $\xi = \beta/(1-\varkappa)$ are given in Fig. 1a. The exact values ρ_2 of the roots of (1.7) were found by numerical solution on a computer. The relative error in the determination of ρ_2 according to Eq. (1.9) does not exceed 1% for $0 \le \xi \le 0.6$ when $\varkappa > 0.2$, and when $\varkappa > 0.6$ the same is true for all values of the core width.

It is apparent from Fig. 1a that the absolute error curves have a parabolic shape, passing through the origin and intercepting the horizontal axis near the point $\xi = 0.5$. It is also a simple matter to determine the ordinates of the points at which the curves terminate for $\xi = 1$. Thus, when the core width β tends to the annular gap width $1 - \varkappa$, the radius ρ_2 tends to unity. Denoting the values of δ_1 at $\xi = 1$ by $c(\varkappa)$, we have

$$c(\varkappa) = \frac{1+\varkappa}{2} - \lambda_0. \tag{1.10}$$

We now consider the curves determined by the expression

$$y = \left(\frac{1+\varkappa}{2} - \lambda_0\right)\xi(2\xi - 1).$$

These curves are parabolas passing through the points $\xi = 0$ and $\xi = 0.5$ and assuming the value $y = c(\varkappa)$ at $\xi = 1$. Consequently, if we form the expression

$$\rho_2^{(1)} = \lambda_0 + \frac{\beta}{2} + \left(\frac{1+\varkappa}{2} - \lambda_0\right)\frac{\beta}{1-\varkappa}\left(\frac{2\beta}{1-\varkappa} - 1\right), \tag{1.11}$$

then the values of ρ_2 determined from this expression will be much closer to the roots of the transcendental equation (1.7) than those determined by Eq. (1.9). Curves of the absolute errors $\delta_2 = \rho_2 - \rho_2^{(1)}$ for various values of $\varkappa$ are given in Fig. 1b. The relative error in the calculation of the root of Eq. (1.7) according to Eq. (1.11) does not exceed 0.55% for $\varkappa = 0.1$, 0.065% for $\varkappa = 0.3$, or 0.009% for $\varkappa = 0.5$. The maximum values $\delta_2^*(\varkappa)$ of the absolute errors for various values of $\varkappa$ are presented in Table 1. It is seen from Table 1 that as $\varkappa$ is increased the error in the calculation of ρ_2 according to Eq. (1.11) falls off sharply. The accuracy with which ρ_2 is determined according to (1.11) is fully acceptable for engineering calculations. The shape of the curves in Fig. 1b suggests a further refinement of Eq. (1.11). The curves in Fig. 1b pass through the points $\xi = 0$ and $\xi = 1$ and have maxima at points whose abscissas are close to $\xi^* = 0.62$. These curves can be approximated by a polynomial of the form

$$y_1 = \delta_2^*(\varkappa)(a_1\xi^3 + b_1\xi^2 + c_1\xi + d_1). \tag{1.12}$$

The constants a_1, b_1, c_1, and d_1 can be determined from the conditions

$$y_1 = 0 \text{ for } \xi = 0 \text{ and } \xi = 1;\ \frac{dy_1}{d\xi} = 0 \text{ for } \xi = \xi^*;\ y_1 = \delta_2^*(\varkappa) \text{ for } \xi = \xi^*. \tag{1.13}$$

Using these conditions, we obtain $a_1 = -4.324$, $b_1 = 2.760$, $c_1 = 1.564$, and $d_1 = 0$. It is seen from Table 1 that the function $\delta_2^*(\varkappa)$ has an exponential-type behavior. For the interval $0.1 \le \varkappa \le 0.5$, in which it is meaningful to use the more accurate expression for ρ_2, the function $\delta_2^*(\varkappa)$ is adequately described by the expression

$$\delta_2^*(\varkappa) = 0.0127 \exp(4.176\varkappa^2 - 12.016\varkappa).$$

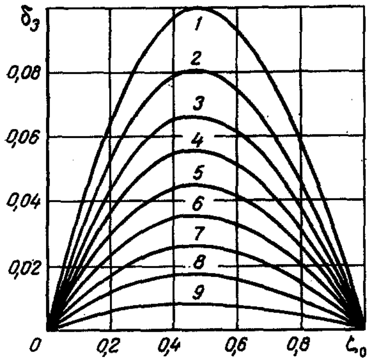

Fig. 2

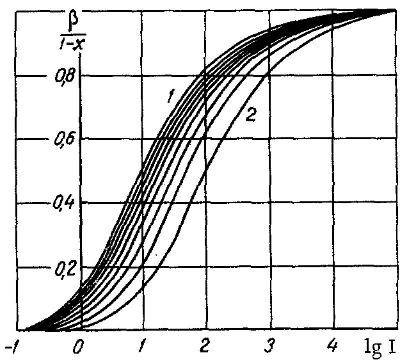

Fig. 3

Fig. 2. Absolute error δ_3 vs ζ_0. 1) $\varkappa$ = 0.1; 2) 0.2; 3) 0.3; 4) 0.4; 5) 0.5; 6) 0.6; 7) 0.7; 8) 0.8; 9) 0.9.

Fig. 3. Relative core width vs log I. 1) $\varkappa$ = 0.1; 2) 0.9.

Injecting the correction (1.12) into expression (1.11), we obtain

$$\rho_2^{(2)} = \lambda_0 + \frac{\beta}{2} + \left(\frac{1+\varkappa}{2} - \lambda_0\right)\frac{\beta}{1-\varkappa}\left(\frac{2\beta}{1-\varkappa} - 1\right)$$

$$+ 0.0127 \exp\,(4.176\varkappa^2 - 12.016\varkappa)\left[-4.324\left(\frac{\beta}{1-\varkappa}\right)^3 + 2.760\left(\frac{\beta}{1-\varkappa}\right)^2 + 1.564\left(\frac{\beta}{1-\varkappa}\right)\right]. \qquad (1.14)$$

In the determination of ρ_2 according to Eq. (1.14) the relative error does not exceed 0.1% for $\varkappa$ = 0.1, 0.015% for $\varkappa$ = 0.3, or 0.0015% for $\varkappa$ = 0.5. Equation (1.14) gives good results for smaller values of $\varkappa$ as well. For example, the relative error is not greater than 1% for $\varkappa$ = 0.02.

2. When the volumetric flow is given, both the core boundary and the core width β (pressure gradient) are unknown quantities. Their determination requires the simultaneous solution of the set of equations (1.4) and (1.7). We transform the flow expression (1.4). We replace P in accordance with (1.5), and the volumetric flow rate by the expression $Q = \pi R_2^2(1 - \varkappa^2)v_{av}$. As a result, Eq. (1.4) is transformed to the following:

$$1 - \varkappa^4 - 2\rho_2(\rho_2 - \beta)\,(1 - \varkappa^2) - \frac{4}{3}(1 + \varkappa^3)\,\beta + \frac{1}{3}(2\rho_2 - \beta)^3\,\beta - \frac{4(1-\varkappa^2)}{I}\,\beta = 0. \qquad (2.1)$$

For a fixed geometry of the annular tube the solution depends on the one dimensionless complex I. As I → 0 we have in the limit the case of a Newtonian fluid flow. The solution of Eq. (2.1) in this case is β = 0, and the solution of Eq. (1.7) is $\rho_2 = \lambda_0$. It is easily verified that in the other limiting case, when I → ∞, the set of equations (1.7) and (2.1) has the solution $\beta = 1 - \varkappa$, $\rho_2 = 1$, i.e., as I → ∞ (v_{av} → 0) the core width tends to the width of the annular gap, and the outside radius of the core tends to the radius of the outside cylinder. Consequently, as I varies from 0 to ∞ the core width varies from zero to the width of the annular gap.

Next we determine approximately the roots of the set of equations comprising (1.7) and (2.1). As shown above, expression (1.11) yields, with good accuracy, the relationship between ρ_2 and β. If we replace ρ_2 in (2.1) in accordance with (1.11), in place of the system of one transcendental and one algebraic equation we obtain one seventh-degree algebraic equation in β. It can be solved by numerical methods. This approach can be used with reasonable accuracy to obtain β and then, from Eq. (1.11), ρ_2. The relative error in the determination of β for all values of I in this case is at most 0.8% for $\varkappa$ = 0.1 or 0.08% for $\varkappa$ = 0.5. The relative error in the determination of ρ_2 according to (1.11) on the basis of the value found for β is even smaller. Still greater accuracy can be obtained by using expression (1.14). The determination of the roots of Eq. (2.1) is far simpler than the determination of the roots of the set of equations, but sizable difficulties are nevertheless encountered. We therefore set ourselves the goal of finding an explicit expression for the roots of the set of equations in terms of the parameters $\varkappa$ and I. We substitute the following expression, rather than (1.11), in place of ρ_2 in Eq. (2.1):

$$\rho_2 = \lambda_0 + \frac{\beta}{2} + c, \qquad (2.2)$$

where c is determined according to (1.10) (we drop the argument $\varkappa$ for notational simplicity). With this substitution Eq. (2.1) goes over to the quadratic equation

$$A\beta^2 - 4B\beta + C = 0, \tag{2.3}$$

where

$$A = \frac{1-\varkappa^2}{2}; \; B = \frac{1+\varkappa^3}{3} - \frac{2}{3}(\lambda_0 + c)^3 + \frac{1-\varkappa^2}{I}; \; C = (1-\varkappa^2)\,[1+\varkappa^2 - 2(\lambda_0 + c)^2].$$

The solution of Eq. (2.3) yields the following zeroth approximation for β:

$$\beta_0 = \frac{2B - \sqrt{4B^2 - AC}}{A}. \tag{2.4}$$

The minus sign is chosen to guarantee that $\beta \to 0$ as $I \to 0$. It is easily verified that $\beta_0 \to 1-\varkappa$ as $I \to \infty$. This result is a consequence of the fact that for $\beta = 1-\varkappa$, according to (2.2), $\rho_2 = 1$, i.e., β and ρ_2 are equal to the exact values of the roots of the set of Eqs. (1.7) and (2.1) when $I = \infty$.

It is readily shown that as $I \to 0$ the quantity β_0 also tends to zero, i.e., to the exact value of the root. Recognizing that $B \to \infty$ as $I \to 0$, we represent the square root in (2.4) as follows:

$$2B\sqrt{1 - \frac{AC}{4B^2}} = 2B\left(1 - \frac{AC}{8B^2} + \ldots\right).$$

Using this expression, we obtain

$$\beta_0 \approx \frac{C}{4B} = \frac{3(1-\varkappa^2)\,[1+\varkappa^2 - 2(\lambda_0+c)^2]\,I}{4\,[3(1-\varkappa^2) + (1+\varkappa^3)\,I - 2(\lambda_0+c)^3\,I]}. \tag{2.5}$$

Hence it is clear that as $I \to 0$ the quantity β_0 also tends to zero. Therefore, we arrive at the conclusion that as I increases from 0 to ∞ the approximate value β_0 increases from 0 to $1-\varkappa$, i.e., varies in the same interval as the exact value β. At the end-points of the interval the approximate value β_0 obtained according to Eq. (2.4) coincides with the exact value, i.e., the absolute error is zero at the end-points of the interval. Curves of the absolute error $\delta_3 = \beta - \beta_0$ as a function of $\zeta_0 = \beta_0/(1-\varkappa)$ for several values of $\varkappa$ are shown in Fig. 2. These curves have one striking attribute: their maxima are located at points whose abscissas are close to $\zeta_0 = 0.47$. This fact facilitates the choice of approximating function. We seek it in the form

$$y_2 = m(\varkappa)\,(\zeta_0^4 + a_2\zeta_0^3 + b_2\zeta_0^2 + c_2\zeta_0 + d_2). \tag{2.6}$$

Here $m(\varkappa)$ represents the values of the maxima of the curves. The function $m(\varkappa)$ is well approximated by the expression

$$m(\varkappa) = [0.088 + 0.035\,(1-\varkappa)^4]\,(1-\varkappa). \tag{2.7}$$

The constants a_2, b_2, c_2, and d_2 are determined from the conditions

$$y_2 = 0 \text{ for } \zeta_0 = 0 \text{ and } \zeta_0 = 1; \; y_2 = m(\varkappa) \text{ for } \zeta_0 = 0.47; \; y_2' = 0 \text{ for } \zeta_0 = 0.47.$$

Using these conditions, we found $a_2 = -0.973$, $b_2 = -4.275$, $c_2 = 4.248$, and $d_2 = 0$. With the correction y_2 we obtain the following approximate equation for β:

$$\beta_1 = \beta_0 + [0.088 + 0.035(1-\varkappa)^4]\,(1-\varkappa)\,[\zeta_0^4 - 0.973\,\zeta_0^3 - 4.275\,\zeta_0^2 + 4.248\,\zeta_0). \tag{2.8}$$

The relative error $(\beta - \beta_1)/\beta$ in the determination of β according to Eq. (2.8) is at most 1% for all $\varkappa > 0.1$ and $\zeta_0 > 0.2$, and as ζ_0 is increased (as I is increased) the accuracy improves. For example, when $\varkappa > 0.4$ $\zeta_0 > 0.5$, the relative error does not exceed 0.1%. With a reduction in ζ_0 the relative error increases, attaining from 2.5% to 3.5% as $\zeta_0 \to 0$ for various values of $\varkappa$ greater than 0.1.

We now deduce an approximate equation that should give better results than Eq. (2.8) for small ζ_0. To do so we need to choose an approximating function that will better describe the curves of Fig. 2 near the origin. We can easily find the angular coefficient of the tangents to the curves through the origin. We have

$$\frac{d\delta_3}{d\zeta_0} = \frac{d(\beta - \beta_0)}{d\beta_0}(1-\varkappa) = \left(\frac{d\beta}{d\beta_0} - 1\right)(1-\varkappa). \tag{2.9}$$

For small values of I the quantity β_0 decreases with I in accordance with (2.5). The character of the decrease of the exact value of β with I can be determined with the help of Eq. (2.1). It is clear from this equation that for small I the value of β is of the order I. In Eq. (2.1) we replace ρ_2 by expression (1.9), which is valid for small β. Rejecting second- and higher-order small terms in Eq. (2.1), we obtain

$$\beta \approx \frac{1}{4}(1+\varkappa^2-2\lambda_0^2)\,\mathrm{I}\ . \tag{2.10}$$

According to (2.10) and (2.5), we have in the limit as $\mathrm{I} \to 0$

$$\frac{d\beta}{d\beta_0} = \frac{1+\varkappa^2-2\lambda_0^2}{1+\varkappa^2-2(\lambda_0+c)^2}.$$

Substituting this expression for the derivative into (2.9) and denoting the angular coefficient of the tangents to the curves at $\zeta_0 = 0$ by $n(\varkappa)$, we obtain

$$n(\varkappa) = \frac{2c(2\lambda_0+c)(1-\varkappa)}{1+\varkappa^2-2(\lambda_0+c)^2}. \tag{2.11}$$

For the initial segments of the curves we seek the approximating function in the form

$$y_3 = m(\varkappa)(\zeta_0^5 + a_3\zeta_0^4 + b_3\zeta_0^3 + c_3\zeta_0^2 + d_3\zeta_0 + e_3). \tag{2.12}$$

We determine the constants a_3, b_3, c_3, d_3, and e_3 from the conditions

$$y_3 = 0 \ \text{for}\ \zeta_0 = 0 \text{and} \zeta_0 = 1;\ y_3 = m(\varkappa),$$
$$y_3' = 0 \ \text{for}\ \zeta_0 = 0.47;\ y_3' = n(\varkappa)\ \text{for}\ \zeta_0 = 0.$$

Using these conditions, we finally obtain

$$\beta_1 = \beta_0 + m(\varkappa)\zeta_0^2(\zeta_0^3 + 18.291\zeta_0^2 - 37.119\zeta_0 + 17.829) + n(\varkappa)\zeta_0(-4.527\zeta_0^2 + 8.782\zeta_0 - 5.255). \tag{2.13}$$

In the calculation of β according to Eq. (2.13) the relative error for $\varkappa > 0.1$ is at most 1% for all values of I. But this equation must be used for $\zeta_0 < 0.2$, whereas for $\zeta_0 > 0.2$ it is required to use Eq. (2.8), because it is simpler than (2.13) and affords greater accuracy than (2.13) for large ζ_0.

Consequently, we propose the following program for finding the roots of the set of Eqs. (1.7) and (2.1). For a given value of I determine β_0 according to Eq. (2.4). If $\beta_0 > 0.2$, calculate β according to Eq. (2.8), but if $\beta < 0.2$, then according to Eq. (2.13). Once β has been found, calculate ρ_2 according to Eq. (1.11). The accuracy in this determination of ρ_2 is higher than the accuracy in the determination of β. For $\varkappa > 0.3$, for example, the relative error is at most 0.1% for all values of I.

3. The relations obtained in the preceding section readily enable us to determine the pressure losses in the flow of a viscoplastic fluid in an annular tube. According to (1.5), we obtain the following expression for the pressure drop per unit length of the tube:

$$\frac{dp}{dz} = \frac{2\tau_0}{R_2\beta}. \tag{3.1}$$

It follows from (3.1) that the determination of the pressure gradient reduces to the determination of the core width β. If the volumetric flow rate is given, we can determine the Il'yushin parameter I. From the known value of I the core width β is determined according to Eq. (2.8) or (2.13), and the pressure gradient according to Eq. (3.1). If the pressure gradient has been found, it is a straightforward task to determine the pressure losses in a tube section of length L. Curves of the dimensionless relative core width as a function of the logarithm of I for $\varkappa$ from 0.1 to 0.9 in steps of 0.1 are shown in Fig. 3.

NOTATION

r, z	are the cylindrical coordinates;
R_1, R_2	are the inside and outside radii of the cylinders;
p	is the pressure;
$P = dp/dz$	is the pressure gradient;
μ	is the plastic viscosity;
τ_0	is the shear strength;

$v = (2\mu / PR_2^2) v_z$	is the dimensionless velocity;
$\rho = r/R_2$	is the dimensionless radial coordinate;
ρ_1, ρ_2	are the dimensionless core boundaries;
$\varkappa = R_1/R_2$	is the ratio of the cylinder radii;
$\beta = 2\tau_0/PR_2$	is the dimensionless parameter (core width);
$I = R_2\tau_0/\mu v_{av}$	is the Il'yushin dimensionless parameter;
v_{av}	is the average velocity over the tube cross section.

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